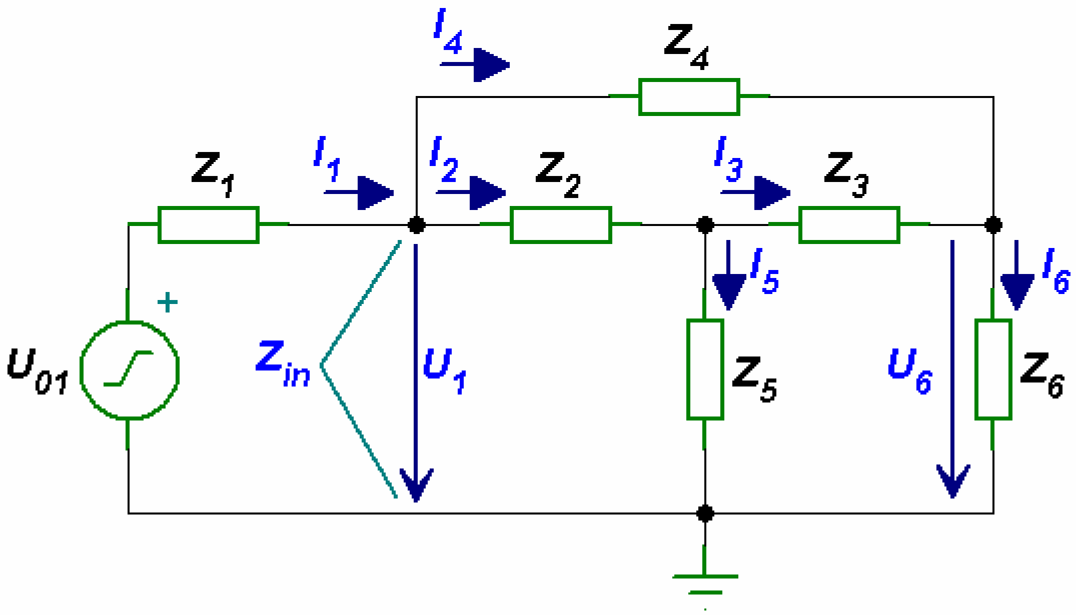
Nodal Analysis - Incidence Matrix A

Example

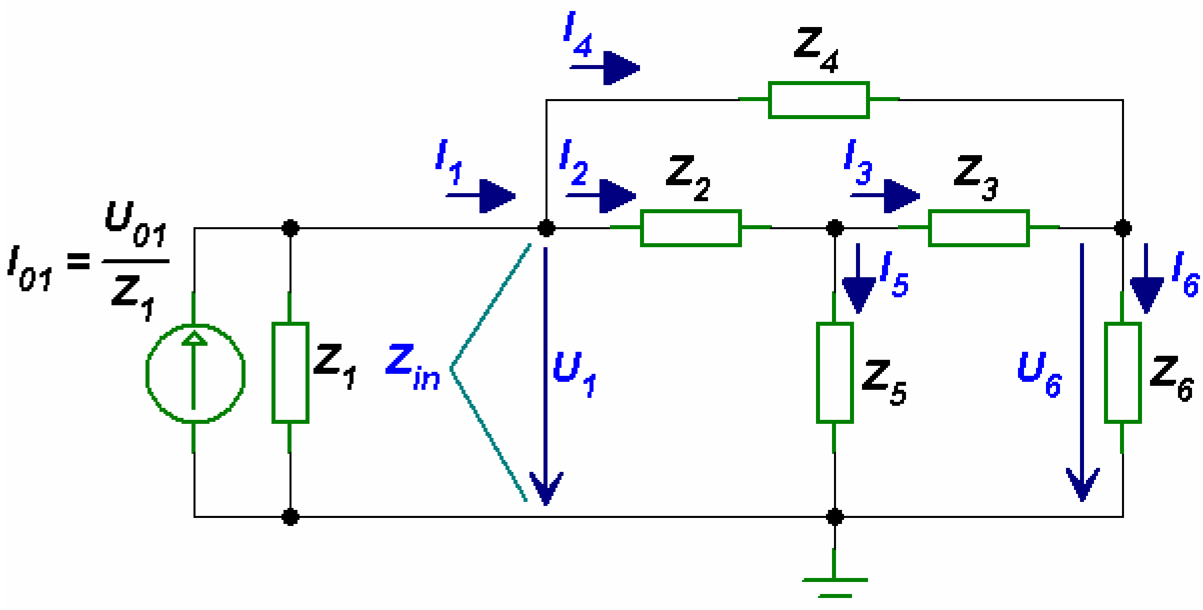
For the circuit shown below find the currents $\boldsymbol{I}_1$ to $\boldsymbol{I}_6$, the voltage $\boldsymbol{U}_6$, the input impedance $\boldsymbol{Z}_{\text{in}}$ and the voltage transfer $\boldsymbol{K}_{06} = \boldsymbol{U}_6 / \boldsymbol{U}_{01}$. Use incidence matrix **A** to solve this task, when $\boldsymbol{Z}_1 = \boldsymbol{Z}_6 = 50\ \Omega$, $\boldsymbol{Z}_2 = \boldsymbol{Z}_3 = -j\ 50\ \Omega$, $\boldsymbol{Z}_4 = \boldsymbol{Z}_5 = j\ 100\ \Omega$, $u_{01}(t) = 10 \cos(\omega t)$ V and $\omega = 10^5$ rad/s. $\boldsymbol{Z}_1$ is the internal resistance of the source.



Circuit with Voltage Source

Solution

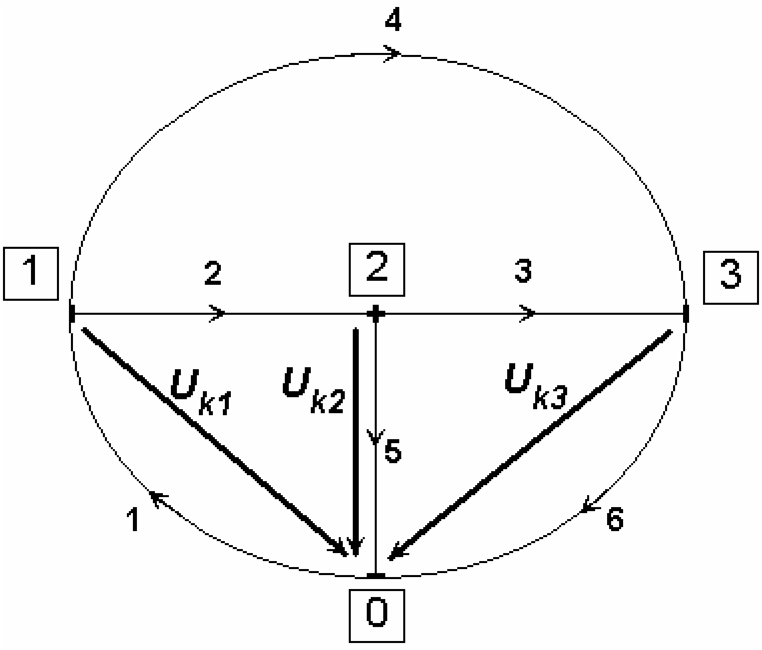
After replacing the voltage source by the current source, we obtain the circuit shown below.



Circuit with Current Source

For the voltage source given by $u_{01}(t) = 10 \cos(\omega t)$ the phasor is $\boldsymbol{U}_{\mathbf{01}} = 10\ \mathrm{e}^{j\,0°} = 10\angle\,0°$. The phasor of the current source is

$$\boldsymbol{I}_{\mathbf{01}} = \frac{\boldsymbol{U}_{\mathbf{01}}}{\boldsymbol{Z}_1} = \frac{10\angle\,0°}{50} = 0.2\angle\,0°\ \text{A}$$



Circuit Diagram

For the circuit diagram shown above we get the incidence matrix **A**. We assume that the currents leaving a node are positive.

$$\mathbf{A} = \overset{\begin{matrix}1 & 2 & 3 & 4 & 5 & 6\end{matrix}}{\begin{bmatrix} -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{bmatrix}}\begin{matrix} \mathit{1} \\ \mathit{2} \\ \mathit{3} \end{matrix}$$

The matrix $\mathbf{I_0}$ of the branch sources and the branch admittance matrix **Y** are

$$\mathbf{I_0} = \begin{bmatrix} \boldsymbol{I}_{\mathbf{01}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{Y} = \begin{bmatrix} 1/\boldsymbol{Z}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/\boldsymbol{Z}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/\boldsymbol{Z}_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\boldsymbol{Z}_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/\boldsymbol{Z}_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/\boldsymbol{Z}_6 \end{bmatrix}$$

The matrix of the nodal current sources is

$$\mathbf{I_{0k}} = \mathbf{A}\ \mathbf{I_0}$$

The nodal admittance matrix is

$$\mathbf{Y_k = A\ Y\ {}^{T}A} \qquad ({}^{T}\mathbf{A} \text{ is the transpose of the matrix } \mathbf{A})$$

The nodal voltage matrix is

$$\mathbf{U_k} = -\,\mathbf{Y_k^{-1}}\ \mathbf{I_{0k}} \qquad (\mathbf{Y_k^{-1}} \text{ is the inverse of the matrix } \mathbf{Y_k})$$

The branch voltage matrix is

$$\mathbf{U} = {}^{T}\mathbf{A}\ \mathbf{U_k}$$

The input impedance is

$$\boldsymbol{Z}_{\mathbf{in}} = \frac{\boldsymbol{U}_1}{\boldsymbol{I}_1} = \frac{\boldsymbol{U}_{\mathbf{k1}}}{\boldsymbol{I}_{01} - \dfrac{\boldsymbol{U}_{\mathbf{k1}}}{\boldsymbol{Z}_1}} \qquad (\boldsymbol{U}_{\mathbf{k1}} / \boldsymbol{Z}_1 + \boldsymbol{I}_1 = \boldsymbol{I}_{01} \quad \Rightarrow \quad \boldsymbol{I}_1 = \boldsymbol{I}_{01} - \boldsymbol{U}_{\mathbf{k1}} / \boldsymbol{Z}_1)$$

The voltage $\boldsymbol{U}_6$ and the voltage transfer $\boldsymbol{K}_{06}$ are

$$\boldsymbol{U}_6 = \boldsymbol{U}_{\mathbf{k3}} \qquad \boldsymbol{K}_{06} = \frac{\boldsymbol{U}_6}{\boldsymbol{U}_{01}} = \frac{\boldsymbol{U}_6}{\boldsymbol{Z}_1 \boldsymbol{I}_{01}}$$

The MATLAB program for solving this task is

MATLAB Script

```
clear; clc
% input values
% the impedances are in ohms
Z1=50; Z2=-j*50; Z3=-j*50; Z4=j*100; Z5=j*100; Z6=50;
% the currents are in amperes
% current i01:
i01max=0.2; i01angle=0; % angle in degrees
% complex representation of the current i01
I01=i01max*exp(j*i01angle*pi/180);
% incidence matrix A
A=[-1  1  0  1  0  0;
    0 -1  1  0  1  0;
    0  0 -1 -1  0  1];
% column matrix I0
I0=[I01; 0; 0; 0; 0; 0];
% diagonal matrix Y
Y=diag([1/Z1 1/Z2 1/Z3 1/Z4 1/Z5 1/Z6]);
I0k=A*I0;
Yk=A*Y*A';
Uk=-inv(Yk)*I0k;
U=A'*Uk
Zin=Uk(1)/(I01-Uk(1)/Z1)
U6=Uk(3)
Ku=U6/(Z1*I01)
```

The results obtained from MATLAB are

```
U =
  -9.5000 - 1.5000i
   3.5000 - 0.5000i
   6.5000 + 0.5000i
  10.0000 - 0.0000i
   6.0000 + 2.0000i
  -0.5000 + 1.5000i

Zin =
  5.0000e+001 +3.0000e+002i

U6 =
  -0.5000 + 1.5000i

Ku =
  -0.0500 + 0.1500i
```